A note on the abelianizations of finite-index subgroups of the mapping class group

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1 Introduction

Let $\Sigma_{g,b}^{p}$ be an oriented genus g surface with b boundary components and p punctures and let $\mathrm{Mod}(\Sigma_{g,b}^{p})$ be its *mapping class group*, that is, the group of isotopy classes of orientation–preserving diffeomorphisms of $\Sigma_{g,b}^{p}$ that fix the boundary components and punctures pointwise (we will omit b or p when they are zero). A long–standing conjecture of Ivanov (see [6] for a recent discussion) says that for $g \geq 3$, the group $\mathrm{Mod}(\Sigma_{g,b}^{p})$ does not virtually surject onto $\mathbb{Z}$. In other words, if Γ is a finite-index subgroup of $\mathrm{Mod}(\Sigma_{g,b}^{p})$, then $\mathrm{H}_1(\Gamma;\mathbb{R}) = 0$.

The goal of this note is to offer some evidence for this conjecture. If G is a group and $g \in G$, then we will denote by $[g]_G$ the corresponding element of $\mathrm{H}_1(G;\mathbb{R})$. Also, for a simple closed curve γ on $\Sigma_{g,b}^{p}$, we will denote by T_γ the corresponding right Dehn twist. Observe that if Γ is any finite-index subgroup of $\mathrm{Mod}_{g,b}^{p}$, then $T_\gamma^n \in \mathrm{Mod}_{g,b}^{p}$ for some $n \geq 1$. Our first result is the following.

Theorem A (Powers of twists vanish)**.** *For some $g \geq 3$, let $\Gamma < \mathrm{Mod}(\Sigma_{g,b}^{p})$ satisfy $[\mathrm{Mod}(\Sigma_{g,b}^{p}) : \Gamma] < \infty$ and let γ be a simple closed curve on $\Sigma_{g,b}^{p}$. Pick $n \geq 1$ such that $T_\gamma^n \in \Gamma$. Then $[T_\gamma^n]_\Gamma = 0$.*

Remark. After this paper was written, Bridson informed us that in unpublished work, he had proven a result about mapping class group actions on CAT(0) spaces that implies Theorem A. Bridson's work will appear in [3].

We use this to verify Ivanov's conjecture for a class of examples. For a long time, the only positive evidence for Ivanov's conjecture was a result of Hain [5] that says that it holds for all finite–index subgroups containing the *Torelli group* $\mathscr{I}_{g,b}^{p}$, that is, the kernel of the action of $\mathrm{Mod}(\Sigma_{g,b}^{p})$ on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ induced by filling in all the punctures and boundary components. The group $\mathscr{I}_{g,b}^{p}$ contains the *Johnson kernel* $\mathscr{K}_{g,b}^{p}$, which is the subgroup generated by Dehn twists about separating curves. A result of Johnson [7] says that $\mathscr{K}_{g,b}^{p}$ is an infinite-index subgroup of $\mathscr{I}_{g,b}^{p}$.

For a subgroup Γ of $\mathrm{Mod}(\Sigma_{g,b}^{p})$, denote by $K(\Gamma)$ the subgroup of $\Gamma \cap \mathscr{K}_{g,b}^{p}$ generated by the set

$$\{T_\gamma^n \mid \gamma \text{ a separating curve, } n \in \mathbb{Z}, \text{ and } T_\gamma^n \in \Gamma\}.$$

If $\mathscr{K}_{g,b}^{p} < \Gamma$, then $K(\Gamma) = \Gamma \cap \mathscr{K}_{g,b}^{p}$, but the converse does not hold. Our second result is the following.

Theorem B (Subgroups containing large pieces of Johnson kernel)**.** *For some $g \geq 3$, let $\Gamma < \mathrm{Mod}(\Sigma_{g,b}^{p})$ satisfy $[\mathrm{Mod}(\Sigma_{g,b}^{p}) : \Gamma] < \infty$. Assume that $[\Gamma \cap \mathscr{K}_{g,b}^{n} : K(\Gamma)] < \infty$. Then $\mathrm{H}_1(\Gamma;\mathbb{R}) = 0$.*

As a corollary, we obtain the following result, which was recently proven by Boggi [2] via a difficult algebro-geometric argument under the assumption $b = p = 0$.

Corollary C (Subgroups containing Johnson kernel)**.** *For some* $g \geq 3$*, let* $\Gamma < \mathrm{Mod}(\Sigma_{g,b}^p)$ *satisfy* $[\mathrm{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$*. Assume that* $\mathscr{K}_{g,b}^n < \Gamma$*. Then* $\mathrm{H}_1(\Gamma; \mathbb{R}) = 0$*.*

Remark. McCarthy [11] proved that Ivanov's conjecture fails in the case $g = 2$.

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2 Notation and basic facts about group homology

If M is a G-module, then M_G will denote the *coinvariants* of the action, that is, the quotient of M by the submodule generated by the set $\{x - g(x) \mid x \in M, g \in G\}$. This appears in the 5-term exact sequence [4, Corollary VII.6.4], which asserts the following. If

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is a short exact sequence of groups, then for any ring R, there is an exact sequence

$$\mathrm{H}_2(G;R) \longrightarrow \mathrm{H}_2(Q;R) \longrightarrow (\mathrm{H}_1(K;R))_Q \longrightarrow \mathrm{H}_1(G;R) \longrightarrow \mathrm{H}_1(Q;R) \longrightarrow 0.$$

If $G_2 < G_1$ are groups satisfying $[G_1 : G_2] < \infty$ and R is a ring, then for all k there exists a *transfer map* of the form $t : \mathrm{H}_k(G_1;R) \to \mathrm{H}_k(G_2;R)$ (see, e.g., [4, Chapter III.9]). The key property of t (see [4, Proposition III.9.5]) is that if $i : \mathrm{H}_k(G_2;R) \to \mathrm{H}_k(G_1;R)$ is the map induced by the inclusion, then $i \circ t : \mathrm{H}_k(G_1;R) \to \mathrm{H}_k(G_1;R)$ is multiplication by $[G_1 : G_2]$. In particular, if $R = \mathbb{R}$, then we obtain a right inverse $\frac{1}{[G_1:G_2]}t$ to i. This yields the following standard lemma.

Lemma 2.1. *Let* $G_2 < G_1$ *be groups satisfying* $[G_1 : G_2] < \infty$*. For all* k*, the map* $\mathrm{H}_k(G_2;\mathbb{R}) \to \mathrm{H}_k(G_1;\mathbb{R})$ *is surjective.*

3 Proof of Theorem A

Let $n \geq 1$ be the smallest integer such that $T_\gamma^n \in \Gamma$.

We first claim that there exists a subsurface $S \hookrightarrow \Sigma_{g,b}^p$ whose genus is at least 2 with the following property. Let $i : \mathrm{Mod}(S) \to \mathrm{Mod}(\Sigma_{g,b}^p)$ be the induced map ("extend by the identity"). Then there exists some boundary component β of S such that $i(T_\beta) = T_\gamma$. There are two cases. If γ is nonseparating, then let S be the complement of a regular neighborhood of γ. Observe that $S \cong \Sigma_{g-1,b+2}^p$, so the genus of S is at least 2. If instead γ is separating, then let S be the component of $\Sigma_{g,b}^p$ cut along γ whose genus is maximal. Since $g \geq 3$, this subsurface must have genus at least 2. The claim follows.

Define $\Gamma' = i^{-1}(\Gamma)$. We have $T_\beta^n \in \Gamma'$, and it is enough to show that $[T_\beta^n]_{\Gamma'} = 0$. Let $\overline{S}$ be the result of gluing a punctured disc to β and let $\overline{\Gamma}'$ be the image of Γ' in $\mathrm{Mod}(\overline{S})$. There is a diagram of central extensions

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\times n"] & \Gamma' \arrow[r] \arrow[d] & \overline{\Gamma}' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Mod}(S) \arrow[r] & \mathrm{Mod}(\overline{S}) \arrow[r] & 1
\end{tikzcd}$$

with $\mathbb{Z} < \mathrm{Mod}(S)$ and $\mathbb{Z} < \Gamma'$ generated by T_β and T_β^n, respectively. The last 4 terms of the corresponding diagram of 5-term exact sequences are

$$\begin{tikzcd}
\mathrm{H}_2(\overline{\Gamma}';\mathbb{R}) \arrow[r, "f_1"] \arrow[d, "f_2"] & \mathbb{R} \arrow[r] \arrow[d, "\cong"] & \mathrm{H}_1(\Gamma';\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{H}_1(\overline{\Gamma}';\mathbb{R}) \arrow[r] \arrow[d] & 0 \\
\mathrm{H}_2(\mathrm{Mod}(\overline{S});\mathbb{R}) \arrow[r, "f_3"] & \mathbb{R} \arrow[r] & \mathrm{H}_1(\mathrm{Mod}(S);\mathbb{R}) \arrow[r] & \mathrm{H}_1(\mathrm{Mod}(\overline{S});\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

We remark that there are no nontrivial coinvariants in these sequences since our extensions are central. We must show that f_1 is a surjection. Since S has genus at least 2, we have $\mathrm{H}_1(\mathrm{Mod}(S);\mathbb{R}) = 0$ (see, e.g., [10]), so f_3 is a surjection. Since $[\mathrm{Mod}(\overline{S}) : \overline{\Gamma}'] < \infty$, Lemma 2.1 implies that f_2 is a surjection, so f_1 is a surjection, as desired.

4 Proof of Theorem B

4.1 Two facts about $\mathrm{Sp}_{2g}(\mathbb{Z})$

We will need two standard facts about finite-index subgroups Γ of $\mathrm{Sp}_{2g}(\mathbb{Z})$, both of which follow from the fact that Γ is a lattice in $\mathrm{Sp}_{2g}(\mathbb{R})$.

For the first, since $\mathrm{Sp}_{2g}(\mathbb{R})$ is a connected simple Lie group with finite center and real rank g, the group Γ has Kazhdan's property (T) when $g \geq 2$ (see, e.g., [13, Theorem 7.1.4]). One standard property of groups with property (T) is that they have no nontrivial homomorphisms to $\mathbb{R}$ (see, e.g., [13, Theorem 7.1.7]). Combining these facts, we obtain the following theorem.

Theorem 4.1. *For some $g \geq 2$, let $\Gamma < \mathrm{Sp}_{2g}(\mathbb{Z})$ satisfy $[\mathrm{Sp}_{2g}(\mathbb{Z}) : \Gamma] < \infty$. Then $\mathrm{H}_1(\Gamma;\mathbb{R}) = 0$.*

For the second, since $\mathrm{Sp}_{2g}(\mathbb{R})$ is a connected noncompact simple real algebraic group, we can apply the Borel density theorem (see, e.g., [13, Theorem 3.2.5]) to deduce that Γ is Zariski dense in $\mathrm{Sp}_{2g}(\mathbb{R})$. This implies that any finite dimensional nontrivial irreducible $\mathrm{Sp}_{2g}(\mathbb{R})$-representation V must also be an irreducible Γ-representation; indeed, if V' was a nontrivial proper Γ-submodule of V, then the subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ preserving V' would be a proper subvariety of $\mathrm{Sp}_{2g}(\mathbb{R})$ containing Γ. Recall that the ring of coinvariants V_Γ of V under Γ is the quotient V/K, where $K = \langle x - g(x) \mid x \in V, g \in \Gamma \rangle$. Since $K \neq 0$, we can apply Schur's lemma to deduce that $K = V$, i.e. that $V_\Gamma = 0$. We record this fact as the following theorem.

Theorem 4.2. *For some $g \geq 1$, let $\Gamma < \mathrm{Sp}_{2g}(\mathbb{Z})$ satisfy $[\mathrm{Sp}_{2g}(\mathbb{Z}) : \Gamma] < \infty$ and let V be a nontrivial irreducible $\mathrm{Sp}_{2g}(\mathbb{R})$-representation. Then $V_\Gamma = 0$.*

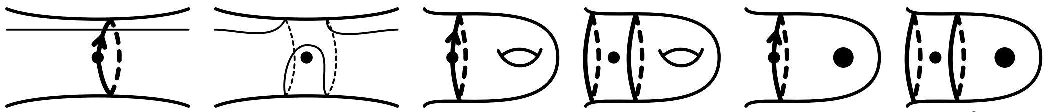

Figure 1: *a–f. Curves needed for proof of Lemma 4.3*

4.2 Two preliminary lemmas

We will need two lemmas. The first is the following, which slightly generalizes a theorem of Johnson [8].

Lemma 4.3. *For $g \geq 3$, we have $\mathscr{I}^p_{g,b}/\mathscr{K}^p_{g,b} \cong (\wedge^3 H)/H \oplus H^{b+p}$, where $H = \mathrm{H}_1(\Sigma_g;\mathbb{Z})$.*

Proof. Since $\mathscr{K}^p_{g,b}$ contains all twists about boundary curves, we can assume that $b = 0$.

Building on work of Johnson [9], Hain [5] proved that

$$\mathrm{H}_1(\mathscr{I}^p_g;\mathbb{R}) \cong (\wedge^3 H_{\mathbb{R}})/H_{\mathbb{R}} \oplus H^p_{\mathbb{R}},$$

where $H_{\mathbb{R}} = \mathrm{H}_1(\Sigma_g;\mathbb{R})$. Also, Johnson [9, Lemma 2] proved that for $x \in \mathscr{K}^p_g$, we have $[x]_{\mathscr{I}^p_g} = 0$ (Johnson only considered the case where $p = 0$, but his argument works in general). It follows that

$$\mathrm{H}_1(\mathscr{I}^p_g/\mathscr{K}^p_g;\mathbb{R}) \cong (\wedge^3 H_{\mathbb{R}})/H_{\mathbb{R}} \oplus H^p_{\mathbb{R}}. \tag{1}$$

We will prove the lemma by induction on p. The base case $p = 0$ is a theorem of Johnson [8]. Assume now that $p > 0$ and that the lemma is true for all smaller p. Fixing a puncture $*$ of Σ^p_g, work of Birman [1] and Johnson [9] gives an exact sequence

$$1 \longrightarrow \pi_1(\Sigma^{p-1}_g,*) \longrightarrow \mathscr{I}^p_g \longrightarrow \mathscr{I}^{p-1}_g \longrightarrow 1,$$

where the map $\mathscr{I}^p_g \to \mathscr{I}^{p-1}_g$ comes from "forgetting the puncture $*$". Quotienting out by $\mathscr{K}^p_g$, we obtain an exact sequence

$$1 \longrightarrow \pi_1(\Sigma^{p-1}_g,*)/(\pi_1(\Sigma^{p-1}_g,*)\cap\mathscr{K}^p_g) \longrightarrow \mathscr{I}^p_g/\mathscr{K}^p_g \longrightarrow \mathscr{I}^{p-1}_g/\mathscr{K}^{p-1}_g \longrightarrow 1.$$

By induction, we have

$$\mathscr{I}^{p-1}_g/\mathscr{K}^{p-1}_g \cong (\wedge^3 H)/H \oplus H^{p-1}.$$

Set $A = \pi_1(\Sigma^{p-1}_g,*)/(\pi_1(\Sigma^{p-1}_g,*)\cap\mathscr{K}^p_g)$. We will prove that A is a quotient of H. We will then be able to conclude that $\mathscr{I}^{p-1}_g/\mathscr{K}^{p-1}_g$ acts trivially on A, so $\mathscr{I}^p_g/\mathscr{K}^p_g$ is the abelian group

$$(\wedge^3 H)/H \oplus H^{p-1} \oplus A.$$

Using (1), a simple dimension count will then imply that A cannot be a proper quotient of H, and the lemma will follow.

The element of $\mathscr{I}^p_g$ corresponding to $\delta \in \pi_1(\Sigma^{p-1}_g,*)$ "drags" $*$ around δ. As shown in Figures 1.a–b, a simple closed curve $\gamma \in \pi_1(\Sigma^{p-1}_g,*)$ corresponds to $T_{\gamma_1}T^{-1}_{\gamma_2} \in \mathscr{I}^p_g$, where γ_1 and γ_2 are the boundary components of a regular neighborhood of γ. In particular, if γ is a simple closed separating curve, then as shown in Figures 1.c–d, the corresponding element of $\mathscr{I}^p_g$ is a product of

separating twists. Since $[\pi_1(\Sigma_g^{p-1},*),\pi_1(\Sigma_g^{p-1},*)]$ is generated by simple closed separating curves (see, e.g., [12, Lemma A.1]), we deduce that $[\pi_1(\Sigma_g^{p-1},*),\pi_1(\Sigma_g^{p-1},*)] \subset \pi_1(\Sigma_g^{p-1},*)\cap\mathscr{K}_g^p$. Thus $A=\pi_1(\Sigma_g^{p-1},*)/(\pi_1(\Sigma_g^{p-1},*)\cap\mathscr{K}_g^p)$ is a quotient of $\mathrm{H}_1(\Sigma_g^{p-1};\mathbb{Z})$. Finally, as shown in Figures 1.e–f, all simple closed curves that are homotopic into punctures are also contained in $\pi_1(\Sigma_g^{p-1},*)\cap\mathscr{K}_g^p$, so we conclude that A is a quotient of $H=\mathrm{H}_1(\Sigma_g;\mathbb{Z})$, as desired. □

For the second lemma, define $Q_{g,b}^p = \mathrm{Mod}_{g,b}^p/\mathscr{K}_{g,b}^p$.

Lemma 4.4. *For some $g \geq 3$, let $Q' < Q_{g,b}^p$ satisfy $[Q_{b,b}^p : Q'] < \infty$. Then $\mathrm{H}_1(Q';\mathbb{R}) = 0$.*

Proof. Restricting the short exact sequence

$$1 \longrightarrow \mathscr{I}_{g,b}^p/\mathscr{K}_{g,b}^p \longrightarrow Q_{b,b}^p \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

to Q', we obtain a short exact sequence

$$1 \longrightarrow B \longrightarrow Q' \longrightarrow \overline{Q}' \longrightarrow 1,$$

where B and $\overline{Q}'$ are finite index subgroups of $\mathscr{I}_{g,b}^p/\mathscr{K}_{g,b}^p$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$, respectively. The last 3 terms of the associated 5-term exact sequence are

$$(\mathrm{H}_1(B;\mathbb{R}))_{\overline{Q}'} \longrightarrow \mathrm{H}_1(Q';\mathbb{R}) \longrightarrow \mathrm{H}_1(\overline{Q}';\mathbb{R}) \longrightarrow 0.$$

By Theorem 4.1, we have $\mathrm{H}_1(\overline{Q}';\mathbb{R})=0$. Letting $H=\mathrm{H}_1(\Sigma_g;\mathbb{Z})$, Lemma 4.3 says that

$$\mathscr{I}_{g,b}^p/\mathscr{K}_{g,b}^p \cong (\wedge^3 H)/H \oplus H^{b+p}.$$

Since B is a finite-index subgroup of $\mathscr{I}_{g,b}^p/\mathscr{K}_{g,b}^p$, we get that B is itself abelian and

$$\mathrm{H}_1(B;\mathbb{R}) \cong B\otimes\mathbb{R} \cong (\mathscr{I}_{g,b}^p/\mathscr{K}_{g,b}^p)\otimes\mathbb{R} \cong (\wedge^3 H_\mathbb{R})/H_\mathbb{R} \oplus H_\mathbb{R}^{b+p},$$

where $H_\mathbb{R}=\mathrm{H}_1(\Sigma_g;\mathbb{R})$. Both $(\wedge^3 H_\mathbb{R})/H_\mathbb{R}$ and $H_\mathbb{R}$ are nontrivial finite-dimensional irreducible representations of $\mathrm{Sp}_{2g}(\mathbb{R})$, so Theorem 4.2 implies that $(\mathrm{H}_1(B;\mathbb{R}))_{\overline{Q}'}=0$, and we are done. □

4.3 The proof of Theorem B

The last 3 terms of the 5-term exact sequence associated to the short exact sequence

$$1 \longrightarrow \Gamma\cap\mathscr{K}_{g,b}^p \longrightarrow \Gamma \longrightarrow \Gamma/(\Gamma\cap\mathscr{K}_{g,b}^p) \longrightarrow 1$$

are

$$(\mathrm{H}_1(\Gamma\cap\mathscr{K}_{g,b}^p;\mathbb{R}))_{\Gamma/(\Gamma\cap\mathscr{K}_{g,b}^p)} \xrightarrow{i} \mathrm{H}_1(\Gamma;\mathbb{R}) \longrightarrow \mathrm{H}_1(\Gamma/(\Gamma\cap\mathscr{K}_{g,b}^p);\mathbb{R}) \longrightarrow 0.$$

By assumption, $[\Gamma\cap\mathscr{K}_{g,b}^p : K(\Gamma)]<\infty$, so Lemma 2.1 implies that the map $\mathrm{H}_1(K(\Gamma);\mathbb{R})\to\mathrm{H}_1(\Gamma\cap\mathscr{K}_{g,b}^p;\mathbb{R})$ is surjective. Since $K(\Gamma)$ is generated by powers of twists, Theorem A allows us to deduce that $i=0$. Also, $\Gamma/(\Gamma\cap\mathscr{K}_{g,b}^p)$ is a finite-index subgroup of $Q_{g,b}^p$, so Lemma 4.4 implies that $\mathrm{H}_1(\Gamma/(\Gamma\cap\mathscr{K}_{g,b}^p);\mathbb{R})=0$, and we are done.

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